Online Appendix to

Statehood, Democracy and Preindustrial Development

by

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Equations in Arabic numerals below refer to those in the original paper. Equations in (small) Roman numerals refer those appearing in these very notes.

1 Public good investment with inequality under non-statehood

Here we find conditions under which an equilibrium exists such that only the richer group, and no other, wants to invest in the public good. In principle, we must consider two cases: when the richer group chooses to establish a state (setting $x_t > 0$) and when it does not (setting $x_t = 0$). However, the former case generates less public good provision by the richer group, so we can focus on that case to derive sufficient conditions under which only the richer group wants invest in the public good.

With income $\widehat{\lambda}Y_t$, investment in extractive capacity equals $x_t = X(\widehat{\lambda}Y_t)$; see (vii). If only the richer group invests in the public good then its investment is given by (10) with $x_t = X(\widehat{\lambda}Y_t)$, i.e., $s_{i,t} = \beta[1 - X(\widehat{\lambda}Y_t)]$. Total public good provision thus equals $K_{t+1} = \sum_{i=1}^{N} K_{i,t+1} = \beta[1 - X(\widehat{\lambda}Y_t)]\widehat{\lambda}Y_t$. For the other groups to not want to invest, it must hold that $\partial V_{i,t}^N / \partial s_{i,t} < 0$ when $s_{i,t} = 0$, $\lambda_{i,t} = (1 - \widehat{\lambda})/(N - 1)$, and $K_{t+1} = \beta[1 - X(\widehat{\lambda}Y_t)]\widehat{\lambda}Y_t$. Using the expression in (A2), this implies

$$\frac{\partial V_{i,t}^N}{\partial s_{i,t}} = -(1-\beta) + \frac{\beta\left(\frac{1-\widehat{\lambda}}{N-1}\right)Y_t}{\beta[1-X(\widehat{\lambda}Y_t)]\widehat{\lambda}Y_t} < 0. \tag{i}$$

Next note from Proposition 2 (b) below that $X'(\widehat{\lambda}Y_t) < 0$, so the inequality in (i) always holds if

$$1-\beta > \frac{\left(\frac{1-\widehat{\lambda}}{N-1}\right)}{[1-X(0)]\widehat{\lambda}} = \left[\frac{1+\beta(1-p)}{N-1}\right]\frac{1-\widehat{\lambda}}{\widehat{\lambda}}, \tag{ii}$$

where the equality follows from $1 - X(0) = 1/[1 + \beta(1 - p)]$; see Proposition 2 (a) below. The condition in (ii) can be rewritten as in (A3) in the paper.

2 Authoritarian statehood

The following proposition characterizes $\Lambda(Y)$ defined in (11) and (12):

Proposition 1 *The function $\Lambda(Y)$ has the following properties:*
(a) $\Lambda(0) = 0$.
(b) $\lim_{Y\to\infty}\Lambda(Y) = 1 - p$.
(c) $\Lambda'(Y) > 0$.
(d) $\Lambda(Y) < 1 - p$ *for all* $Y < \infty$.
(e) $\Lambda'(0) = \frac{\beta(1-p)}{1+\beta(1-p)} \in (0,1)$.

Proof:

Part (a): From (12) follows that $a(0) = 1 - \beta p/(1+\beta) \in (0,1)$ and $b(0) = 0$. Then (11) implies that $\Lambda(0) = (1/2)\left[a(0) - \sqrt{[a(0)]^2}\right] = 0$.

Part (b): The solution in (11) and (12) is derived from the polynomial in (A9), which can be written

$$\left[\Lambda(Y_t^R)\right]^2 - a(Y_t^R)\Lambda(Y_t^R) + b(Y_t^R) = 0. \tag{iii}$$

Dividing (iii) by Y_t^R, using (12), and letting $\lambda^* = \lim_{Y\to\infty}\Lambda(Y)$, we see that

$$\begin{aligned} &\lim_{Y\to\infty}\left(\tfrac{[\lambda^*]^2}{Y}\right) - \lim_{Y\to\infty}\left(\tfrac{a(Y)}{Y}\right)\lambda^* + \lim_{Y\to\infty}\tfrac{b(Y)}{Y} \\ &= 0 - \left(\tfrac{\beta}{1+\beta}\right)\lambda^* + \left(\tfrac{\beta}{1+\beta}\right)(1-p) = 0, \end{aligned} \tag{iv}$$

implying that $\lambda^* = \lim_{Y\to\infty}\Lambda(Y) = 1 - p$.

Part (c): Implicitly differentiating (iii) we find that

$$\Lambda'(Y_t^R) = \left(\frac{\beta}{1+\beta}\right)\left[\frac{1}{a(Y_t^R) - 2\Lambda(Y_t^R)}\right]\left[1 - p - \Lambda(Y_t^R)\right] > 0, \tag{v}$$

where the inequality follows from $a(Y_t^R) - 2\Lambda(Y_t^R) = \sqrt{[a(Y_t^R)]^2 - 4b(Y_t^R)} > 0$, and part (a) of Proposition 1: since $\Lambda(0) = 0$, and $\Lambda'(Y_t^R)$ approaches zero when $\Lambda(Y_t^R)$ approaches $1-p$, we know that $\Lambda(Y_t^R)$ cannot exceed $1-p$ for any finite Y_t^R.

Part (d): The claim follows from parts (b) and (c).

Part (e): Using $\Lambda(0) = 0$ [see part (a)], $a(0) = [1 + \beta(1-p)]/(1+\beta)$ [see (12)], and (v) it follows that

$$\Lambda'(0) = \left(\frac{\beta}{1+\beta}\right)\left[\frac{1+\beta}{1+\beta(1-p)}\right](1-p) = \frac{\beta(1-p)}{1+\beta(1-p)} \in (0,1), \tag{vi}$$

where we use $\beta \in (0,1)$ and $p \in (0,1)$.

Q.E.D.

Next we summarize how x_t depends on Y_t^R. Let

$$x_t = \frac{\Lambda(Y_t^R)}{Y_t^R} \equiv X(Y_t^R). \tag{vii}$$

The properties of $X(Y)$ can be summarized as follows:

Proposition 2 *The function $X(Y) = \Lambda(Y)/Y$ has the following properties:*
(a) $\lim_{Y\to 0} X(Y) = \frac{\beta(1-p)}{1+\beta(1-p)} \in (0,1)$.
(b) $X'(Y) < 0$.
(c) $\lim_{Y\to\infty} X(Y) = 0$.

Proof:

Part (a): Using l'Hôpital's rule, it follows that $\lim_{Y\to 0} X(Y) = \lim_{Y\to 0}[\Lambda(Y)/Y] = \lim_{Y\to 0}\Lambda'(Y) = \Lambda'(0)$. Part (e) of Proposition 1 finishes the proof.

Part (b): Using (A7) and setting $\lambda_{t+1}^R = x_t Y_t^R = \Lambda(Y_t^R)$ and $x_t = X(Y_t^R)$ gives

$$\frac{X(Y_t^R)}{1 - X(Y_t^R)} = \beta(1-p) - \beta p \left(\frac{\Lambda(Y_t^R)}{1-\Lambda(Y_t^R)}\right). \tag{viii}$$

It follows that $X'(Y_t^R)$ and $\Lambda'(Y_t^R)$ have opposite signs. Part (c) of Proposition 1 then finishes the proof.

Part (c): Using part (b) of Proposition 1 it follows that $\lim_{Y\to\infty} X(Y) = \lim_{Y\to\infty}[\Lambda(Y)/Y] = (1-p)\lim_{Y\to\infty}(1/Y) = 0$.

Q.E.D.

The properties of the function $\Gamma(Y_t^R)$ are described by the following proposition:

Proposition 3 *The function $\Gamma(Y)$ has the following properties:*
(a) $\Gamma(0) = 0$.
(b) $\lim_{Y\to 0}\Gamma'(Y) = 0$.
(c) $\lim_{Y\to\infty}\Gamma'(Y) = \beta Z(1-p) > 1$.
(d) $\frac{\partial[\Gamma(Y)/Y]}{\partial Y} > 0$.
(e) There exists a unique $Y^* > 0$, such that $Y^* = \Gamma(Y^*)$.

Proof:

Part (a): The claim follows from (14) and part (a) of Proposition 1.

Part (b): Use (14) to see that

$$\Gamma'(Y) = \beta Z\left\{[1-\Lambda'(Y)]\Lambda(Y) + [Y - \Lambda(Y)]\Lambda'(Y)\right\}. \tag{ix}$$

Then the claim follows from applying parts (a) and (e) of Proposition 1 to (ix).

Part (c): Note that

$$\lim_{Y\to\infty}\Gamma'(Y)=\lim_{Y\to\infty}\frac{\Gamma(Y)}{Y}=\lim_{Y\to\infty}\beta Z\left[1-X(Y)\right]\Lambda(Y)=\beta Z(1-p)>1, \tag{x}$$

where we have used l'Hôpital's rule, (14), $X(Y)=\Lambda(Y)/Y$, part (b) of Proposition 1, and part (c) of Proposition 2; the inequality follows from Assumption 2.

Part (d): Note that

$$\frac{\partial}{\partial Y}\left[\frac{\Gamma(Y)}{Y}\right]=\frac{\partial\left\{\beta Z\left[1-X(Y)\right]\Lambda(Y)\right\}}{\partial Y}>0, \tag{xi}$$

where the inequality follows from $X'(Y)<0$ and $\Lambda'(Y)>0$; recall part (c) of Proposition 1 and part (b) of Proposition 2.

Part (e): Let $R(Y)=\Gamma(Y)/Y$, so that any Y^* must be such that $R(Y^*)=1$. Parts (a) and (c) of this proposition, together with l'Hôpital's rule, say that $\lim_{Y\to 0}R(Y)=0$ and $\lim_{Y\to\infty}R(Y)>1$, and part (d) says that $R'(Y)>0$. Thus, there exists a unique Y^* such that $R(Y^*)=1$.

Q.E.D.

3 Ranking the three systems

3.1 Democratic statehood vs. non-statehood

To compare democratic statehood to non-statehood, we begin by showing the following:

Lemma 4 *$\omega<1/N$.*

Proof: First rewrite ω in (21) as

$$\omega=\frac{1}{N}\left[N\left(\frac{1-\widehat{\lambda}}{N-1}\right)^{\left(1-\frac{1}{N}\right)}\left(\widehat{\lambda}\right)^{\frac{1}{N}}\right]^q=\frac{1}{N}\left[\frac{\left(1-\widehat{\lambda}\right)^{\left(1-\frac{1}{N}\right)}\left(\widehat{\lambda}\right)^{\frac{1}{N}}}{\frac{1}{N}\left(N-1\right)^{\left(1-\frac{1}{N}\right)}}\right]^q. \tag{xii}$$

It is straightforward to show that the numerator of the expression within square brackets in (xii) is maximized at $\widehat{\lambda}=1/N$, and that

$$\max_{\lambda\in[0,1]}\left\{(1-\lambda)^{\left(1-\frac{1}{N}\right)}\lambda^{\frac{1}{N}}\right\}=\left(1-\frac{1}{N}\right)^{\left(1-\frac{1}{N}\right)}\left(\frac{1}{N}\right)^{\frac{1}{N}}=\frac{1}{N}\left(N-1\right)^{\left(1-\frac{1}{N}\right)}. \tag{xiii}$$

From (xiii) follows that the expression in square brackets in (xii) is strictly less than one for $\widehat{\lambda}\neq 1/N$, implying that $\omega<1/N$ for all $q>0$.

Q.E.D.

Next we show the following:

Proposition 4 *There exists a* $\overline{\lambda} < 1$*, given by (26), such that, for any* $Y_t > 0$*,* $V^D(Y_t) > V^N(\lambda_t Y_t)$*, if and only if,* $\lambda_t < \overline{\lambda}$*.*

Proof: Setting $V^D(Y_t) = V^N(\overline{\lambda} Y_t)$, and using (23), (21) and (25), gives $\ln(\overline{\lambda}) = -\ln(N) - \beta \ln(\omega)$, which gives (26). Then $\overline{\lambda} < 1$ is shown in Section 3.4 below. The claim of the proposition follows from $V^N(\overline{\lambda} Y_t)$ being increasing $\overline{\lambda}$.

Q.E.D.

3.2 Autocratic statehood vs. non-statehood

To compare autocratic statehood to non-statehood, we begin by showing the following:

Proposition 5 $\Psi(Y)$ *has the following properties:*
(a) $\Psi'(Y) > 0$*.*
(b) $\lim_{Y\to 0} \Psi(Y) = -\infty$*.*
(c) $\lim_{Y\to \infty} \Psi(Y) = \beta \ln\left[(1-p)^{1-p} p^p\right]$*.*

Proof:

Part (a): From (22) follows that

$$\Psi'(Y) = \frac{-X'(Y)}{1-X(Y)} + \beta \left[\frac{1-p}{\Lambda(Y)} - \frac{p}{1-\Lambda(Y)}\right] \Lambda'(Y) > 0, \tag{xiv}$$

where the inequality follows from $X'(Y) < 0$, $\Lambda'(Y) > 0$, and $\Lambda(Y) < 1-p$; see Propositions 1 and 2.

Part (b): Note from part (a) of Proposition 1 that $\lim_{Y\to 0} = \ln\left[\Lambda(Y)\right] = -\infty$, since $\Lambda(0) = 0$. Propositions 1 and 2 also imply that the other terms are finite, since $\lim_{Y\to 0} \ln[1 - X(Y)] = -\ln[1 + \beta(1-p)]$ and $\ln\left[1 - \Lambda(0)\right] = \ln(1) = 0$.

Part (c): The claim follows from part (e) of Proposition 1, part (c) of Proposition 2, and some algebra.

Q.E.D.

We can now state the following:

Proposition 6 *There exists a* $\overline{Y} > 0$*, such that* $V^R(\lambda_t Y_t) > V^N(\lambda_t Y_t)$ *if, and only if,* $\lambda_t Y_t > \overline{Y}$*.*

Proof: The net gain to the decisive group from choosing authoritarian statehood over non-statehood can be written

$$H(\lambda_t Y_t) \equiv V^R(\lambda_t Y_t) - V^N(\lambda_t Y_t) = \Psi(\lambda_t Y_t) - \beta p \ln(N-1) - \beta \ln(\omega). \tag{xv}$$

Recall from Proposition 5 that $\Psi'(Y) > 0$, $\lim_{Y\to 0}\Psi(Y) = -\infty$, and $\lim_{Y\to\infty}\Psi(Y) = \beta\ln[(1-p)^{1-p}p^p]$. Thus, it is straightforward to see that $H'(Y) > 0$ and $\lim_{Y\to 0} H(Y) = -\infty$.

Next, from (xv) and some algebra follows that

$$\begin{aligned} \lim_{Y\to\infty} H(Y) &= \lim_{Y\to\infty}\Psi(Y) - \beta p\ln(N-1) - \beta\ln(\omega) \\ &> \beta\ln[(1-p)^{1-p}p^p] - \beta p\ln(N-1) + \beta\ln(N) = \beta\ln\left[\tfrac{(1-p)^{1-p}p^p}{F(N)}\right], \end{aligned} \tag{xvi}$$

where the first inequality uses $\omega < 1/N$ (see Lemma 4), and where we have used the notation in (xix), $F(N) = (N-1)^p/N$. As shown in Section 3.4, $F(N) \leq (1-p)^{1-p}p^p$ for all $N \geq 1$. Thus, the expression in square brackets on the right-hand side in (xvi) is greater than one, so $\lim_{Y\to\infty} H(Y) > 0$.

Since $H'(Y) > 0$, $\lim_{Y\to 0} H(Y) = -\infty$, and $\lim_{Y\to\infty} H(Y) > 0$ there exists some finite $\overline{Y} > 0$ such that $H(\overline{Y}) = 0$.

Q.E.D.

3.3 Democratic vs. autocratic statehood

Finally, the comparison of autocratic and democratic statehood is summarized by the following proposition:

Proposition 7 *(a) There exists a* $\underline{Y} > 0$ *such that* $V^D(Y) > V^R(\lambda Y)$ *for all* $\lambda \in (0,1]$ *and* $Y < \underline{Y}$.
(b) $V^D(Y) > V^R(\lambda Y)$ *for all* (λ, Y) *such that* $\lambda \in (0, \underline{\lambda}]$ *and* $Y > 0$.
(c) There exists a function $\phi : (\underline{Y}, \infty) \to (\underline{\lambda}, 1)$*, such that:*
(i) $V^D(Y) = V^R(\phi(Y)Y)$*;*
(ii) $\phi'(Y) < 0$*; and*
(iii) $\lim_{Y\to\infty}\phi(Y) = \underline{\lambda}$.

Proof: Let the net gain to the decisive group from choosing authoritarian statehood over democracy be

$$G(\lambda_t, Y_t) \equiv V^R(\lambda_t Y_t) - V^D(Y_t) = \Psi(\lambda_t Y_t) + \ln(\lambda_t) - \beta p\ln(N-1) + \ln(N). \tag{xvii}$$

Part (a): Recall from Proposition 5 that $\Psi'(Y) > 0$, $\lim_{Y\to 0}\Psi(Y) = -\infty$, and $\lim_{Y\to\infty}\Psi(Y) = \beta\ln[(1-p)^{1-p}p^p]$. This implies that $G(1,Y) = \Psi(Y) - \beta p\ln(N-1) + \ln(N)$ is such that $G_Y(1,Y) > 0$ and $\lim_{Y\to 0} G(1,Y) = -\infty$. Also, from (27) and some algebra follows that $\lim_{Y\to\infty} G(1,Y) = -\ln(\underline{\lambda}) > 0$. Thus, there exists some $\underline{Y} > 0$ such that $G(1,\underline{Y}) = 0$. Since

$G(\lambda, Y)$ is decreasing in both its arguments, it follows that $G(\lambda, Y) < 0$, for all (λ, Y) such that $Y < \underline{Y}$ and $\lambda < 1$.

Part (b): (i) From part (a) and $G_Y(1, Y) > 0$ follows that $G(1, Y) > 0$ for any $Y > \underline{Y}$. Since $G_\lambda(\lambda, Y) > 0$, and $\lim_{\lambda \to 0} G(\lambda, Y) = -\infty$, there must exist some $\lambda \in (0, 1)$ such that $G(\lambda, Y) = 0$ for all $Y > \underline{Y}$.

(ii) Note from (xvii) and Proposition 5 (a) that $G_\lambda(\lambda, Y) > 0$ and $G_Y(\lambda, Y) > 0$ (where the subscripts indicate the variables with respect to which G is differentiated). Implicit differentiation shows that $\phi'(Y) = -G_\lambda(\lambda, Y)/G_Y(\lambda, Y) < 0$.

(iii) Setting $\lim_{Y \to \infty} G(\lambda, Y) = \beta \ln\left[(1-p)^{1-p} p^p\right] + \ln(\lambda) - \beta p \ln(N-1) + \ln(N) = 0$, and solving for λ, gives (27).

Q.E.D.

3.4 Showing that $1/N < \underline{\lambda} < \overline{\lambda} < 1$

Let

$$q^{\max} = \left(\frac{1-\beta}{\beta}\right) \frac{\ln(N)}{\ln\left[\frac{1}{N}\left(\frac{N-1}{1-\widehat{\lambda}}\right)^{1-\frac{1}{N}}\left(\frac{1}{\widehat{\lambda}}\right)^{\frac{1}{N}}\right]} > 0, \tag{xviii}$$

where the inequality follows from the expression in square brackets in (xviii) being minimized and equal to one at $\widehat{\lambda} = 1/N$ (see the proof of Lemma 4) and thus always greater than one for $\widehat{\lambda} > 1/N$. It is now easily seen from (26) that $q < q^{\max}$ implies $\overline{\lambda} < 1$. From (26), (27), and $1/\omega > N$ (see Lemma 4) a sufficient condition for $\overline{\lambda} > \underline{\lambda}$ to hold is that

$$F(N) \equiv \frac{(N-1)^p}{N} \leq (1-p)^{1-p} p^p. \tag{xix}$$

It can be shown that $\max_{N \geq 0} F(N) = F(1/[1-p]) = (1-p)^{1-p} p^p$. Thus, the inequality in (xix) must hold, implying that $\overline{\lambda} > \underline{\lambda}$.

Finally, to show that $\underline{\lambda} > 1/N$ we note that the expression in square brackets in (27) is greater than one. This follows from $N \geq 2$ and the denominator being less than one, since $p \in (0, 1)$.