Some mathematics preparation for Econ 2450

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Course mostly about using economic models

Requires a bit of math to understand

Nothing that should be news

What follows is a useful summary

Numbers

Natural numbers: $1, 2, 3...$

Integers: $... - 2, -1, 0, 1, 2...$

Real numbers: "all" numbers that lie on the real line (e.g. 5, -100, $\pi = 3.14...$, $\frac{1}{11}$)

Rational numbers: numbers that can be written as $\frac{a}{b}$ where a and b are integers

Irrational numbers: real numbers that are not rational numbers

Inequalities

$a > b$ means a is strictly greater than b

$a \geq b$ means a is weakly greater than (i.e., greater than or equal to) b

Double inequalities, intervals

If $a > b$ and $a < c$ we can write

$$b < a < c$$

or

$$a \in (b, c)$$

Means a lies on the (open) interval (b, c)

If $a \geq b$ and $a \leq c$ we can write

$$b \leq a \leq c$$

or

$$a \in [b, c]$$

Means a lies on the (closed) interval $[b, c]$

If $a > b$ and $a \leq c$ we can write

$$b < a \leq c$$

or

$$a \in (b, c]$$

Means a lies on the (half-open) interval $(b, c]$

Functions

Here: mostly single-variable functions

What are functions?

- Functions are "rules"
- Functions describe how one variable depends on another variable
- Same as everyday speak: "the grade is a function of how much you study..."

Graphs are used to illustrate functions in diagrams

Definitions

A function $f(x)$ of a real variable x with *domain* D_f is a rule which assigns a *unique* real number to each number in D_f

D_f can be defined by a simple statement, e.g. $D_f = [0, 1]$

The set of values that $f(x)$ takes as x varies over D_f is called the *range* of f, and is denoted R_f

All functions in this course will be given by *formulas*, e.g.

$$\begin{aligned} f(x) &= 2x^2 \\ f(x) &= 3x - 14 \\ f(x) &= b + ax \end{aligned}$$

where a and b are constants, i.e., do not depend on the variable x

Exponential and logarithmic functions

Power functions take the form

$$f(x) = a^{bx}$$

where a and b are constants

a is the *base*; bx is the *exponent*

If $a = e$, where e is defined from

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828,$$

then we get an *exponential* function:

$$f(x) = e^{bx}$$

The (natural) *logarithmic* function

$$f(x) = \ln(x)$$

is defined from

$$e^{\ln(x)} = x$$

Note that:

- $e^{bx} > 0$ for all x
- $\ln(x)$ is defined only for $x > 0$

Some important equalities

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\ln(\tfrac{x}{y}) = \ln(x) - \ln(y)$$

$$\ln(x^a) = a\ln(x)$$

$$e^{x+y} = e^x e^y$$

$$(e^x)^y = e^{xy}$$

Derivatives

Many functions can be *differentiated*

To differentiate a function is the same as taking the *derivative* of the function (but not same as e.g. deriving a function)

The derivative of $f(x)$ is (usually) denoted $f'(x)$ and defined from

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

On notation

There are many ways to denote derivatives

We often denote $f(x)$ by some some other variable y, and write $y = f(x)$

We can then write $f'(x)$ as any of the following:

$$y'$$

$$\frac{\partial y}{\partial x}$$

$$\frac{\partial f(x)}{\partial x}$$

$$\frac{dy}{dx}$$

$$\frac{df(x)}{dx}$$

The notation $\partial f(x)/\partial x$ is usually referred to as the *partial* derivative of $f(x)$ with respect to x

The notation $df(x)/dx$ is usually referred to as the *total* derivative of $f(x)$ with respect to x

When $f(x)$ as only one argument (i.e., x) the total and partial derivatives mean the same thing

Differentiating some common functions

$$f(x) = x^2 \Rightarrow f'(x) = 2x$$

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$$

$$f(x) = e^{ax} \Rightarrow f'(x) = ae^{ax}$$

$$f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x} = x^{-1}$$

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = \frac{-1}{x^2} = -x^{-2}$$

Some rules

The chain rule

$$f(x) = g\left[h(x)\right] \Rightarrow f'(x) = g'\left[h(x)\right] h'(x)$$

The product rule

$$f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + g(x)h'(x)$$

A special case of the product rule: let

$$f(x) = \frac{g(x)}{k(x)} = g(x)h(x),$$

where $h(x) = 1/k(x)$; then

$$\begin{aligned} f'(x) &= g'(x)\overbrace{\left[\frac{1}{k(x)}\right]}^{h(x)} + g(x)\overbrace{\left(-\left[\frac{1}{k(x)}\right]^2 k'(x)\right)}^{h'(x)} \\ &= \frac{g'(x)k(x) - g(x)k'(x)}{[k(x)]^2} \end{aligned}$$

Some other useful tricks:

$$\frac{f'(x)}{f(x)} = \frac{\partial}{\partial x}\left[\ln(f(x))\right]$$

$$f(x) = g(x)h(x) \Rightarrow \frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)} + \frac{h'(x)}{h(x)}$$

Optimization

Consider the problem of finding some x [in the domain of $f(x)$] that maximizes $f(x)$

That x is called a (global) maximum point of $f(x)$

Often denoted x^*, defined from

$$f(x^*) \geq f(x) \text{ for all } x \in D_f$$

If the graph is $f(x)$ is shaped like an inverted U the maximum point is easy enough to see; draw to illustrate

Often we have a formula for $f(x)$ and want to find the maximum point in terms of the constants (parameters) of $f(x)$

The first-order condition

The maximum point of $f(x)$ must satisfy certain conditions

One such condition states that the derivative of $f(x)$ equals zero when evaluated at the maximum point: called the *first-order condition (FOC)*

Example: let $f(x) = x(1 - x)$, where $D_f = [0, 1]$

Task: find the maximum point of $f(x)$

Find $f'(x)$, using product rule

$$f'(x) = 1 - x + x(-1) = 1 - 2x$$

Setting $f'(x^*) = 1 - 2x^* = 0$ gives

$$x^* = \frac{1}{2}$$

The second-order condition

Sometimes the FOC gives not a maximum point but a minimum point

Example: $f(x) = \frac{x^2}{2} - x$; $f'(x) = 0$ gives $x = 1$, which is a minimum point

How can we check if the FOC gives a minimum or a maximum?

- In this course the problems are (almost always) rigged so that the FOC gives whatever you are looking for (the minimum or maximum point)

- More generally, we can look at the second derivative of $f(x)$, denoted $f''(x)$

- Called *second-order condition (SOC)*

- SOC states that:

  * For x^* to be a (local) maximum it must hold that $f''(x^*) < 0$

  * For x^* to be a (local) minimum it must hold that $f''(x^*) > 0$

Implicit differentiation

Sometimes we do not have an explicit expression for $f(x)$

Instead, $f(x)$ may be defined *implicitly*

Example: let y be a function of x defined from

$$G(x, y) = 0$$

Task: find dy/dx

Think of both $G(x, y)$ and y as functions of x

Differentiate both sides with respect to x using chain rule

$$\frac{\partial G(x, y)}{\partial x} + \frac{\partial G(x, y)}{\partial y}\frac{dy}{dx} = 0$$

Solve for dy/dx

$$\frac{dy}{dx} = -\frac{\frac{\partial G(x,y)}{\partial x}}{\frac{\partial G(x,y)}{\partial y}}$$

Called *implicit differentiation*

Note:

- If $\partial G(x,y)/\partial x$ and $\partial G(x,y)/\partial y$ have the same sign, then $dy/dx < 0$

- If $\partial G(x,y)/\partial x$ and $\partial G(x,y)/\partial y$ have different signs, then $dy/dx > 0$

“Kinked” functions (involving curly brackets)

Often we want to make sure that a function has a particular range

Example:

- $f(x)$ denotes a probability and is linear in x
- $D_f = \mathcal{R}_+ = [0, \infty)$
- Must make sure R_f falls within $[0, 1]$
- Solution: to “cut off” the function so that $f(x) \leq 1$ for all $x \in D_f$

$$f(x) = \begin{cases} ax & \text{if } x \leq 1/a \\ 1 & \text{if } x > 1/a \end{cases}$$

  Draw graph with kink at $x = 1/a$

Statistics and probability theory

Let x be a (discrete) stochastic variable

Can take N different values: x_1, x_2,..., x_N

Associated probabilities:
$\pi_1 =$ probability that x takes value x_1
$\pi_2 =$ probability that x takes value x_2
...
$\pi_N =$ probability that x takes value x_N

Must hold that probabilities sum up to one:

$$\pi_1 + \pi_2 + ... + \pi_N = \sum_{i=1}^{N} \pi_i = 1$$

$E(x)$ denotes the *mean*, or *expected value*, of x

Definitions of mean and variance:

$$E(x) = \pi_1 x_1 + \pi_2 x_2 + ... + \pi_N x_N = \sum_{i=1}^{N} \pi_i x_i$$

$$V(x) = \pi_1 [x_1 - E(x)]^2 + \pi_2 [x_2 - E(x)]^2$$

$$+ ... + \pi_N [x_N - E(x)]^2$$

$$= \sum_{i=1}^{N} \pi_i [x_i - E(x)]^2$$

$$= E\left[\{x - E(x)\}^2 \right]$$

Useful to note that we can develop expression in square brackets to write variance as

$$V(x) = \left(\pi_1 x_1^2 + \pi_2 x_2^2 + ... + \pi_N x_N^2\right) - [E(x)]^2$$

$$= \left(\sum_{i=1}^{N} \pi_i x_i^2\right) - [E(x)]^2$$

$$= E(x^2) - [E(x)]^2$$

If x and y are both stochastic:

$$\begin{aligned} E(x+y) &= E(x)+E(y) \\ E(x-y) &= E(x)-E(y) \end{aligned}$$

If a is a (non-stochastic) constant:

$$\begin{aligned} E(a) &= a \\ E(a+x) &= a+E(x) \\ E(ax) &= aE(x) \end{aligned}$$

$$\begin{aligned} V(a) &= 0 \\ V(a+x) &= V(a)+V(x)=V(x) \\ V(ax) &= a^2V(x) \end{aligned}$$