Econ 5011 - Midterm Exam
9 February 2005

Problem 1. Consider the Solow model with Cobb-Douglas production, where capital per effective worker evolves according to

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t).$$

The notation is completely standard: total capital is $K(t)$; $k(t) = K(t)/[A(t)L(t)]$ is capital per effective worker; $A(t)$ is efficiency units per worker; $L(t)$ is the number of workers; s is the rate of saving; δ is the depreciation rate; g is the growth rate of $A(t)$; and n the growth rate of $L(t)$. Total output is $Y(t)$, and output per effective worker is given by $f(k(t)) = y(t) = Y(t)/[A(t)L(t)]$. Consumption per effective worker is given by $c(t) = (1 - s)f(k(t))$. At some time $\widetilde{t}$ the growth rate of $A(t)$ (i.e., g), rises from g_0 to g_1, where $0 < g_0 < g_1 < \infty$. To get full score on (a) to (c) below *you do not need to write any equations, or explain anything; just draw the graphs correctly.*

(a) Draw a graph showing the time path of $k(t)$. [3 marks]

(b) Draw a graph showing the time path of $c(t)$. [3 marks]

(c) Draw a graph showing the time path of log output per worker, $\ln[Y(t)/L(t)]$. [4 marks]

Problem 2. Consider a Ramsey model, where population is constant ($n = 0$) and there is no technological progress ($g = 0$), and no depreciation ($\delta = 0$).
Utility is given by

$$U = \int_{t=0}^{\infty} e^{-\rho t} \ln[c(t)]dt,$$

where ρ is the utility discount rate, $c(t)$ is per-capita consumption, and $\ln[c(t)]$ is the instantaneous utility function.
The budget constraint on "flow" form is given by:

$$\dot{k}(t) = w + rk(t) - c(t), \tag{*}$$

where w and r are here constant and exogenous.
Setting initial assets, $k(0)$, to zero the present-value form of the budget constraint becomes:

$$\int_{t=0}^{\infty} e^{-rt} c(t)dt = \int_{t=0}^{\infty} e^{-rt} w dt.$$

The (present-value) Hamiltonian associated with maximizing U subject to (*) becomes:

$$H(c(t), k(t), \lambda(t), t) = e^{-\rho t} \ln[c(t)] + \lambda(t)[w + rk(t) - c(t)].$$

The optimality conditions are: $H_c(\cdot) = 0$, and $H_k(\cdot) = -\dot{\lambda}(t)$ (plus a transversality condition which we do not need for this exercise).

(a) Show how to find an expression for $\dot{c}(t)/c(t)$ is terms of exogenous parameters (i.e., the Euler equation), using the Hamiltonian and the optimality conditions provided. [4 marks]
(b) Find an expression for $c(t)$ in terms of w, ρ, r, and t. [4 marks]
(c) Assume that $r > \rho$. It can then be shown that there exists a t^*, such that $c(t) > w$ for all $t > t^*$. Find t^* in terms of r and ρ. [2 marks]

Problem 3. Consider a Diamond model where consumption in working age ($C_{1,t}$) and old age ($C_{2,t+1}$), are perfect complements, so that utility is given by:

$$U_t = \min\{C_{1,t}, C_{2,t+1}\}$$

Agents choose consumption in old and working age so as to maximize U_t subject to

$$C_{2,t+1} = [w_t A_t - C_{1,t}](1 + r_{t+1}),$$

where $w_t = f(k_t) - f'(k_t)k_t$ is the wage per effective worker; $k_t = K_t/[A_t L_t]$ is the period-t capital stock per effective worker; K_t is total capital in period t; A_t is efficiency units per worker in period t; $r_{t+1} = f'(k_{t+1})$ is the interest on saving held from period t to $t+1$ (there is no depreciation); and L_t is the working population in period t.
Capital accumulation is given by:

$$K_{t+1} = S_t L_t$$

where $S_t = w_t A_t - C_{1,t}$.
(a) Find optimal S_t as a function of $w_t A_t$ and r_{t+1}. [5 marks]
(b) Assume Cobb-Douglas production, $f(k_t) = k_t^{\alpha}$, and let $A_{t+1} = (1 + g)A_t$, and $L_{t+1} = (1 + n)L_t$. It can be seen that there is a level $\overline{\alpha}$, such that $\alpha < \overline{\alpha}$ must hold for a strictly positive steady state level of k_t to exist. Find an expression for $\overline{\alpha}$ in terms of n and g. [5 marks]

Problem 1: see figures below. To see where the answer to (c) comes from, let $z(t) = Y(t)/L(t) = A(t)f(k(t))$. Then $\dot{z}(t)/z(t) = g + \alpha(k)\left\{\dot{k}(t)/k(t)\right\}$, where $\alpha(k) = f'(k)k/f(k) \in (0,1)$. To see what happens to the slope of $\ln[z(t)]$ at $\widetilde{t}$, keep $k(t)$ fixed and differentiate with respect to g. Using the expression for $\dot{k}(t)$ we see that this is increasing in g.

Problem 2: (a)

$$H_c(\cdot) = e^{-\rho t}\left(\frac{1}{c(t)}\right) - \lambda(t) = 0$$

$$H_k(\cdot) = \lambda(t)r = -\dot{\lambda}(t) \quad \Rightarrow \quad \frac{\dot{\lambda}(t)}{\lambda(t)} = -r$$

Using the above we get:

$$e^{-\rho t}\left(\frac{1}{c(t)}\right) = \lambda(t)$$

$$-\rho t - \ln[c(t)] = \ln[\lambda(t)]$$

$$-\rho - \frac{\dot{c}(t)}{c(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)} = -r$$

$$\frac{\dot{c}(t)}{c(t)} = r - \rho$$

(b) The differential equation $\dot{c}(t) = (r-\rho)\,c(t)$ derived under (a) can be solved as $c(t) = c(0)e^{(r-\rho)t}$. Using the present-value budget constraint we get:

$$\int_{t=0}^{\infty} c(t)e^{-rt} = \int_{t=0}^{\infty} e^{-rt} w dt$$

$$c(0)\int_{t=0}^{\infty} e^{-rt}e^{(r-\rho)t} = w\int_{t=0}^{\infty} e^{-rt}dt$$

$$c(0)\int_{t=0}^{\infty} e^{-\rho t} = w\int_{t=0}^{\infty} e^{-rt}dt$$

$$\frac{c(0)}{-\rho}\left[\overbrace{e^{-\rho\times\infty}}^{=0} - \overbrace{e^{-\rho\times 0}}^{=1}\right] = \frac{w}{-r}\left[\overbrace{e^{-r\times\infty}}^{=0} - \overbrace{e^{-r\times 0}}^{=1}\right]$$

$$\frac{c(0)}{\rho} = \frac{w}{r}$$

$$c(0) = \frac{\rho w}{r}$$

which can be substituted into $c(t) = c(0)e^{(r-\rho)t}$; this gives:

$$c(t) = \frac{\rho w}{r} e^{(r-\rho)t}$$

(c) Using the answer under (b), setting $c(t^*) = w$, we see that

$$\tfrac{w\rho}{r} e^{(r-\rho)t^*} = w$$

$$\ln(\tfrac{\rho}{r}) + (r-\rho)t^* = 0$$

$$t^* = \tfrac{\ln(r)-\ln(\rho)}{r-\rho}$$

Problem 3:
(a) The utility function is such that $C_{1,t} = C_{2,t+1}$ must hold. Using the budget constraint, and $S_t = w_t A_t - C_{1,t}$, this gives:

$$C_{2,t+1} = S_t(1+r_{t+1}) = w_t A_t - S_t. = C_{1,t}$$

Solving for S_t we get

$$S_t = \frac{w_t A_t}{2+r_{t+1}}.$$

(b) Cobb-Douglas production implies that $w_t = (1-\alpha)k_t^\alpha$, and $r_{t+1} = \alpha k_{t+1}^{\alpha-1}$. Using the answer under (a) gives:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{(1-\alpha)k_t^\alpha}{2+\alpha k_{t+1}^{\alpha-1}} = \frac{D k_t^\alpha}{2+\alpha k_{t+1}^{\alpha-1}}$$

where

$$D = \frac{1-\alpha}{(1+n)(1+g)}$$

A steady state $k^* > 0$ (if such exists) would thus be given by:

$$k^* = \tfrac{D(k^*)^\alpha}{[2+\alpha(k^*)^{\alpha-1}]}$$

$$k^*\left[2+\alpha(k^*)^{\alpha-1}\right] = D(k^*)^\alpha$$

$$2k^* + \alpha(k^*)^\alpha = D(k^*)^\alpha$$

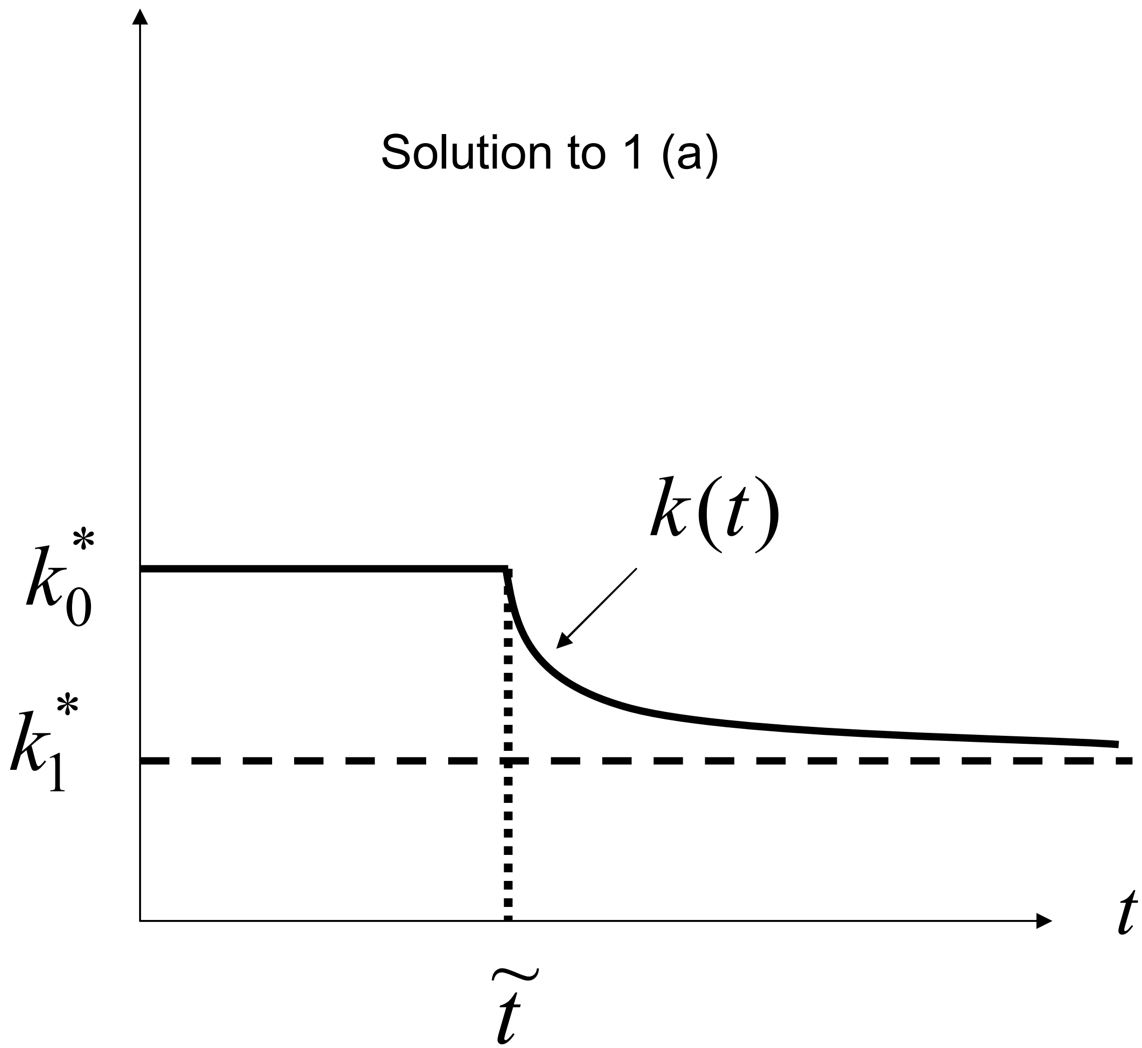
$$2k^* = [D-\alpha](k^*)^\alpha.$$

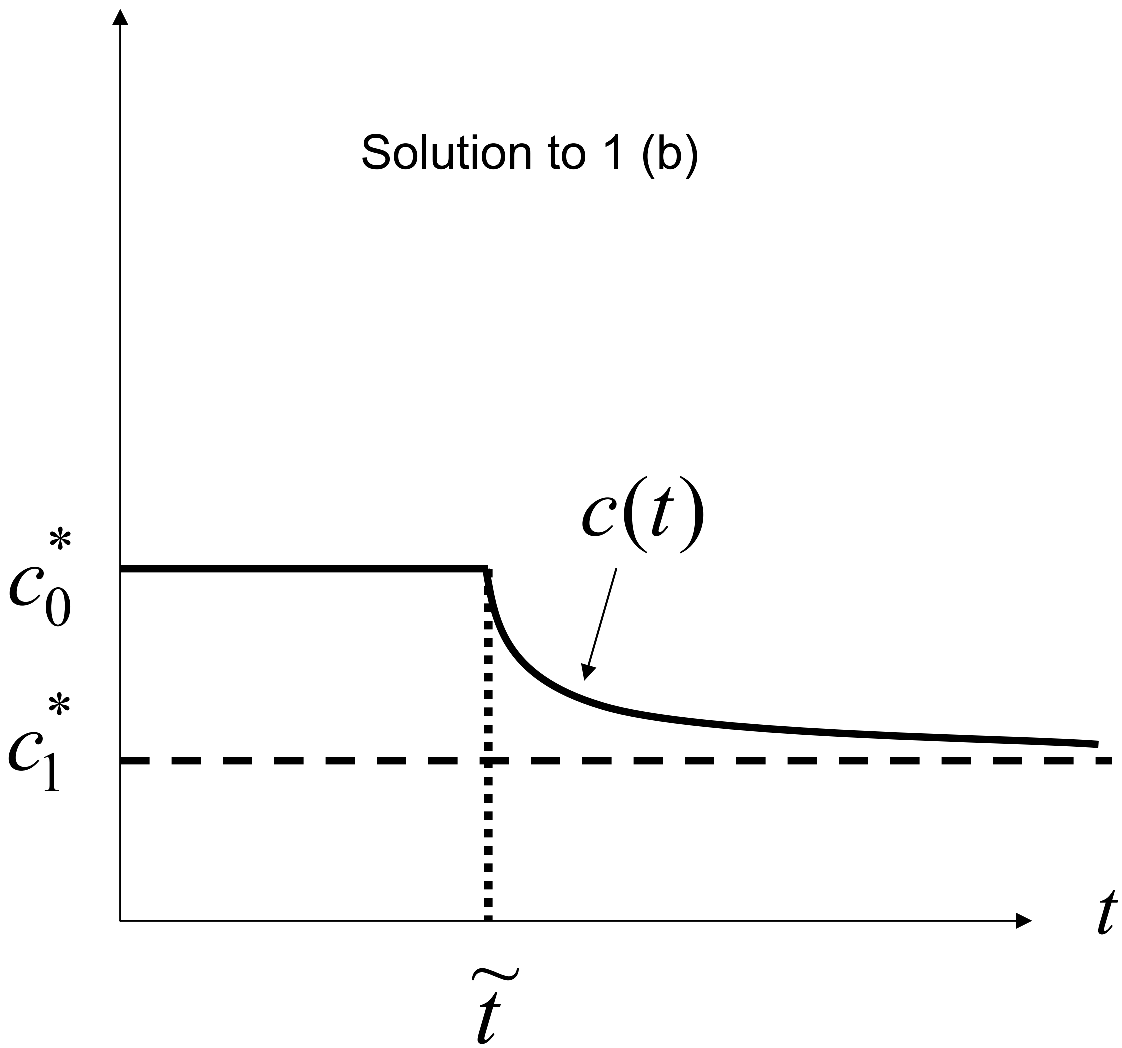
For this equality to hold for some strictly positive k^*, $D-\alpha$ must be positive. That is:

$$D = \frac{1-\alpha}{(1+n)(1+g)} > \alpha$$

or

$$\alpha < \frac{1}{(1+g)(1+n)+1} \equiv \overline{\alpha}$$


Solution to 1 (a)
k_0^*
k_1^*
$k(t)$
t
$\tilde{t}$



Solution to 1 (b)
$c(t)$
c_0^*
c_1^*
t
$\tilde{t}$

Solution to 1 (c)